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LETTER TO THE EDITOR

Path integral formalism for Osp(1/2, *R*) coherent states

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Abstract. A path integral formulation in the representation of coherent states for the non-compact supergroup Osp(1/2, *R*) is introduced. An expression for the transition amplitude connecting two Osp(1/2, *R*) coherent states is constructed, and the corresponding canonical equations of motion derived. A set of generalized Poisson brackets is introduced and interpreted.

Both path integrals [1] and coherent states [2, 3] have played major roles in the study of quantum mechanical systems, particularly for establishing the correspondence between classical and quantum physics. Coherent states for the group SU(2) were introduced by Radcliffe [4] later, the construction of coherent states was generalized to arbitrary Lie groups by Perelomov [5] and Gilmore [6]. More recently, generalized coherent states for supergroups have begun to be investigated [7, 8].

The use of coherent states to provide an alternative method of obtaining the phase space path integral, and hence Hamilton's equations of motion, was pioneered by Klauder and others [9]. This technique has recently been extended to include a formulation in terms of generalized coherent states for SU(1, 1) [10], SU(2) [11] and the *n*-dimensional Euclidean group [12]. The coherent state path integral formalism has also found application in the theoretical study of Berry's geometrical phase [13]. Essential in the formulation of any path integral formalism is the construction of an invariant measure of integration for the coherent states. In [7], we presented a general method for constructing this invariant measure for the coherent states of the non-compact supergroups Osp(1/2*N*, *R*) and studied the representations of Osp(1/2, *R*). In the past few years, there have been hints of physically realized supersymmetry in nuclear [14], atomic [15] and many-body physics [15]. It is therefore a worthwhile effort to study path integral for supergroups.

In this letter, we wish to extend the previous path integral formalism of Sp(2, *R*) coherent states studied in [10] to the non-compact supergroup Osp(1/2, *R*), which contains Sp(2, *R*) as a subgroup. We introduce the non-compact superalgebra Osp(1/2, *R*) next, where we also present a brief summary of the results for the associated coherent states. Later, we present the path integral formulation of the transition amplitude between two Osp(1/2, *R*) coherent states and derive the classical equations of motion for the system. Finally we summarize our work and discuss future extensions and applications of our results.

In our notation, the non-compact superalgebra Osp(*M*/2*N*, *R*) contains the subalgebra O(*M*) that acts on the fermionic space and the subalgebra Sp(2*N*, *R*) which

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acts on the bosonic space. The Osp(1/2, *r*) supergroup has five infinitesimal generators [17] whose defining commutation and anti-commutation relations are given by the following:

$$[K_0, K_\pm] = \pm K_\pm \qquad [K_+, K_-] = -2K_0 \qquad [K_0, V_\pm] = \pm\tfrac{1}{2}V_\pm \tag{1a}$$

$$[K_\pm, V_\pm] = 0 \qquad [K_\pm, V_\mp] = \mp V_\pm \tag{1b}$$

$$\{V_\pm, V_\pm\} = K_\pm \qquad \{V_+, V_-\} = K_0. \tag{1c}$$

The superalgebra Osp(1/2, *R*) clearly contains an subalgebra Sp(2, *R*) spanned by $K_\pm$ and K_0.

We consider only the positive discrete series representations of Osp(1/2, *R*), so the coherent states are constructed as

$$|\alpha, \theta> = \mathcal{N} \exp(\alpha K_+ + \theta V_+)|\tau \qquad k = \tau, m = k\rangle \tag{2}$$

where α is a complex variable and θ is a Grassmann variable. In (2), the quantum numbers τ and k label the irreducible representations of Osp(1/2, *R*) and Sp(2, *R*) respectively and m labels the eigenvalue of K_0. The normalization constant $\mathcal{N}$ ensures that the coherent state is normalized to unity and is given by

$$\mathcal{N} = \left\{\frac{1}{(1-|\alpha|^2)}\left[1 - \frac{1}{2}\frac{\bar{\theta}\theta}{(1-|\alpha|^2)}\right]\right\}^{-\tau} \tag{3}$$

where $\bar{\theta}$ and α^* are the complex conjugates of θ and α. Similarly the overlap between two Osp(1/2, *R*) coherent states can be written

$$\langle \alpha_1, \theta_1|\alpha_2, \theta_2\rangle = \mathcal{N}_1\mathcal{N}_2\left\{\frac{1}{(1-\alpha_2\alpha_1^*)}\left[1 - \frac{1}{2}\frac{\bar{\theta}_1\theta_2}{(1-\alpha_1\alpha_2^*)}\right]\right\}^{-2\tau}. \tag{4}$$

In [7], we constructed the measure of integration and decomposition of unity for Osp(1/2, *R*) coherent states. We found

$$\int \mathrm{d}\mu(\alpha, \theta)|\alpha, \theta\rangle\langle\alpha, \theta|$$

$$= \sum_m |\tau, \tau, m\rangle\langle\tau, \tau, m| + \sum_m |\tau, \tau+\tfrac{1}{2}, m\rangle\langle\tau, \tau+\tfrac{1}{2}, m| = 1 \tag{5a}$$

where

$$\mathrm{d}\mu(\alpha, \theta) = \frac{2}{\pi}\,\mathrm{d}\bar{\theta}\,\mathrm{d}\theta\,\mathrm{d}\alpha^2\left\{\frac{1}{(1-|\alpha|^2)}\left[1 - \frac{1}{2}\frac{\bar{\theta}\theta}{(1-|\alpha|^2)}\right]\right\}^{-\tau}. \tag{5b}$$

In the above integrations over Grassmann variables, we fix our normalization as

$$\int \mathrm{d}\bar{\theta}\,\mathrm{d}\theta(1, \bar{\theta}, \theta) = 0 \qquad \int \mathrm{d}\bar{\theta}\,\mathrm{d}\theta\,\theta\bar{\theta} = 1. \tag{6}$$

It is easy to verify that the Osp(1/2, *R*) coherent states so constructed are 'closest to classical' in the sense of Perelomov [5].

Consider a Hamiltonian, *H*, that is constructed from the generators of the supergroup. The propagator from the coherent state at time t_2 to the coherent state at time t_1 is given by

$$\mathcal{T} \equiv T(\alpha_1, \theta_1, t_1; \alpha_2, \theta_2, t_2) = \left\langle \alpha_1, \theta_1 \middle| \exp\left\{-\frac{\mathrm{i}}{\hbar}H(t_1 - t_2)\right\} \middle| \alpha_2, \theta_2\right\rangle. \tag{7}$$

In principle, we should use a time-ordered exponential to allow for the Hamiltonian being time dependent; however, the modifications needed are straightforward and are omitted here. As usual, we divide the time interval $\Delta T = t_1 - t_2$ into n equal parts. $\varepsilon = \Delta T/n$, and finally take the limit as $n \to \infty$. Thus we find that

$$\mathcal{T} = \lim_{n\to\infty} \left\langle \alpha_1, \theta_1 \left| \left\{ 1 - \frac{\mathrm{i}}{\hbar} H\varepsilon \right\}^n \right| \alpha_2, \theta_2 \right\rangle. \tag{8}$$

The identity operator (equation (5a)) is inserted into each of the equal time internals, leading to the expression

$$\begin{aligned}\mathcal{T} &= \lim_{n\to\infty} \int \ldots \int \prod_k \mathrm{d}\mu(\alpha_k, \theta_k) \prod_k \left\langle \alpha_k, \theta_k \left| \left\{ 1 - \frac{\mathrm{i}}{\hbar} H\varepsilon \right\} \right| \alpha_{k-1}, \theta_{k-1} \right\rangle \\ &= \lim_{n\to\infty} \int \ldots \int \prod_k \mathrm{d}\mu(\alpha_k, \theta_k) \prod_k \langle \alpha_k, \theta_k | \alpha_{k-1}, \theta_{k-1} \rangle \\ &\quad \times \prod_k \left\{ 1 - \frac{\mathrm{i}}{\hbar} \varepsilon \langle \alpha_k, \theta_k | H | \alpha_{k-1}, \theta_{k-1} \rangle / \langle \alpha_k, \theta_k | \alpha_{k-1}, \theta_{k-1} \rangle \right\}. \end{aligned} \tag{9}$$

Here the endpoints are $t_n = t_1$ and $t_0 = t_2$.

The term in the curly bracket in (9) is the simplest to handle, and to first order in ε it can be replaced by the exponential of the expectation value of the Hamiltonian. We now calculate the product term $\Pi \langle \alpha_k, \theta_k | \alpha_{k-1}, \theta_{k-1} \rangle$. The expression for the overlap of the coherent states (4) is inserted into this expression and we find

$$\begin{aligned}&\prod_k \langle \alpha_k, \theta_k | \alpha_{k-1}, \theta_{k-1} \rangle \\ &\quad = \exp\left\{ \sum_k \varepsilon \frac{1}{\varepsilon} \ln\langle \alpha_k, \theta_k | \alpha_{k-1}, \theta_{k-1} \rangle \right\} \\ &\quad = \exp \sum_k \left\{ \frac{\varepsilon\tau}{(1-|\alpha_k|^2)} \left[\left(\alpha_k \frac{\Delta\alpha_k^*}{\varepsilon} - \frac{\Delta\alpha_k}{\varepsilon} \alpha_k^* \right) \left[1 + \frac{1}{2} \frac{\bar\theta_k \theta_k}{(1-|\alpha_k|)} \right] \right.\right. \\ &\qquad \left.\left. + \frac{1}{2} \left(\frac{\Delta\bar\theta_k}{\varepsilon} \theta_k - \bar\theta_k \frac{\Delta\theta_k}{\varepsilon} \right) \right] + \mathcal{O}(\Delta^2) \right\} \\ &\quad \to \exp \int \mathrm{d}t \left\{ \frac{\tau}{(1-|\alpha|^2)} \left[(\alpha\alpha'^* - \alpha^*\alpha') \right.\right. \\ &\qquad \left.\left. + \tfrac{1}{2}(\bar\theta'\theta - \bar\theta\theta') \right] \left[1 + \frac{1}{2} \frac{\bar\theta\theta}{(1-|\alpha|^2)} \right] \right\} \end{aligned} \tag{10}$$

where the prime denote time differentiation and $\Delta\alpha_k = \alpha_k - \alpha_{k-1}$, $\Delta\theta_k = \theta_k - \theta_{k-1}$ and we have neglected terms second order in Δ.

The transition amplitude can then be written in the following formal manner:

$$\mathcal{T} = \int \mathcal{D}\mu(\alpha(t), \theta(t)) \exp\left(\frac{\mathrm{i}}{\hbar} \mathcal{S} \right) \tag{11a}$$

with

$$\mathcal{S} = \int \mathcal{L}(\alpha(t), \alpha^*(t), \alpha'(t), \alpha'^*(t), \theta(t), \bar\theta(t), \theta'(t), \bar\theta(t), t)\, \mathrm{d}t. \tag{11b}$$

$\mathcal{S}$ is the action and the quantity $\mathcal{L}$ is identified as the Lagrangian. The explicit form of the Lagrangian is

$$\mathcal{L}=\left\{\frac{\mathrm{i}\tau}{(1-|\alpha|^2)}[(\alpha\alpha'^*-\alpha^*\alpha')+\tfrac{1}{2}(\bar{\theta}'\theta-\bar{\theta}\theta')]\left(1+\frac{1}{2}\frac{\bar{\theta}\theta}{(1-|\alpha|^2)}\right)\right\}-\mathcal{A} \tag{12a}$$

with

$$\mathcal{H}=\langle\alpha,\theta|H|\alpha,\theta\rangle. \tag{12b}$$

To arrive at the classical limit, we consider the case $\mathcal{S}\ll\hbar$. The dominant contribution to the transition amplitude then comes from where the variation of the action vanishes. Setting $\delta\mathcal{S}=0$, we find the following system of equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\alpha'}-\frac{\partial\mathcal{L}}{\partial\alpha}=0 \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\alpha^{*\prime}}-\frac{\partial\mathcal{L}}{\partial\alpha^*}=0 \tag{13a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\theta'}-\frac{\partial\mathcal{L}}{\partial\theta}=0 \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\bar{\theta}'}-\frac{\partial\mathcal{L}}{\partial\bar{\theta}}=0. \tag{13b}$$

Inserting the expression for the Lagrangian into (13), we obtain the equations of motion for the system:

$$\alpha^{*\prime}=\mathcal{B}\{\alpha^*,\mathcal{H}\}_B-\tfrac{1}{2}\alpha^*\theta\mathcal{F}\{\bar{\theta},\mathcal{H}\}_F \tag{14a}$$

$$\alpha'=\mathcal{B}\{\alpha,\mathcal{H}\}_B+\tfrac{1}{2}\alpha\bar{\theta}\mathcal{F}\{\theta,\mathcal{H}\}_F \tag{14b}$$

$$\bar{\theta}'=-\frac{\alpha\bar{\theta}}{(1-|\alpha|^2)}\mathcal{B}\{\alpha,\mathcal{H}\}_B-\mathcal{F}\{\bar{\theta},\mathcal{H}\}_F \tag{14c}$$

$$\theta'=-\frac{\alpha^*\theta}{(1-|\alpha|^2)}\mathcal{B}\{\alpha,\mathcal{H}\}_B-\mathcal{F}\{\theta,\mathcal{H}\}_F \tag{14d}$$

where

$$\mathcal{B}=\frac{\mathrm{i}(1-|\alpha|^2)^2}{\hbar\tau}\left\{1-\frac{1}{2}\frac{\bar{\theta}\theta}{(1-|\alpha|^2)}\right\} \tag{15a}$$

$$\mathcal{F}=\frac{\mathrm{i}(1-|\alpha|^2)}{\hbar\tau}\left\{1-\frac{1}{2}\frac{\bar{\theta}\theta}{(1-|\alpha|^2)}\right\}. \tag{15b}$$

In (14), we have introduced the Poisson brackets for complex and Grassmann variables:

$$\{A,B\}_B=\left\{\frac{\partial A}{\partial\alpha^*}\frac{\partial B}{\partial\alpha}-\frac{\partial A}{\partial\alpha}\frac{\partial B}{\partial\alpha^*}\right\} \tag{16a}$$

$$\{A,B\}_F=\left\{\frac{\partial A}{\partial\bar{\theta}}\frac{\partial B}{\partial\theta}-\frac{A\overleftarrow{\partial}}{\partial\bar{\theta}}\frac{B\overleftarrow{\partial}}{\partial\theta}\right\}. \tag{16b}$$

Finally, we note that by calculating the exterior derivative of the differential form of the portion of the Lagrangian containing the time derivatives, the equation of motion have the form expected from [10, 11].

In this letter we have introduced a path integral formalism for the non-compact supergroup Osp(1/2, R). The resulting equations of motion contain two Poisson brackets—one for the complex variable α and one for the Grassmann variable θ. The form of the equations of motion follows from the fact that the coherent states were constructed from a supergroup. The existence of the fermion (boson) Poisson bracket in the equations of motion for the bosonic (fermionic) variables is a direct consequence of the supergroup structure.

We have not presented an example here, as we have yet to find a system that exhibits an Osp(1/2, R) symmetry. Work has begun on a coherent state formalism for Osp(2/2, R), where a physically relevant system exists [14]. Results for this will be presented elsewhere.

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